Realization of associative products in terms of Moyal and tomographic symbols

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Abstract

The quantizer-dequantizer method allows to construct associative products on any measure space. Here we consider an inverse problem: given an associative product is it possible to realize it within the quantizer-dequantizer framework? The answer is positive in finite dimensions and we give a few examples in infinite dimensions.

1 Introduction

The standard formulation of quantum mechanics associates pure states with the state vector $|\psi\rangle$ [1] or wave function $\psi(x)$ [2] while density states are

described by density operators [3] [4] or density matrices. Observables are described by Hermitian operators $\hat{A} = \hat{A}^\dagger$ and their statistics, encoded by their highest moments $\langle \hat{A}^n \rangle$, is given by the pairing of the state and the observable $\langle \hat{A}^n \rangle = \mathrm{Tr}[\hat{\rho}\hat{A}^n]$ [3, 4]. On the other hand in classical statistical mechanics on phase-space the state is associated with a probability density $f(q,p)$ and observables are functions $A(q,p)$ on this space. The statistics encoded by the highest moments $\langle A^n \rangle$ is described by the formulae of standard probability theory $\langle A^n \rangle = \int f(q,p) A^n(q,p) dq dp$. An attempt to find a formulation of quantum mechanics which would proceed along the same rules of classical statistics gave origin to the Weyl-Wigner formalism. Here operator symbols [5] are used as observables according to the Weyl map $A(q,p) \to \hat{A}$. The Wigner function [6] $W(q,p)$ which is the Weyl symbol of the density operator $\hat{\rho} \to W(q,p)$ has been introduced and used to describe quantum states. Since operators form a noncommutative $C^\star$–algebra the Weyl map provides a $C^\star$–algebra of functions in the phase space with a noncommutative product–rule called "star-product". The general construction of star–products was considered in many papers (see [7, 8, 9]). In connection with the recently introduced tomographic picture of quantum mechanics (see [10, 11]) the star-product general scheme was discussed in detail [12] where the notion of quantizer and dequantizer operators was introduced for arbitrary star product scheme. Mathematical aspects in abstract form of the star–product in phase space were considered in [13, 14, 15]. In the formulation of the star-product approach [12] one considers operators $\hat{D}(x)$ and $\hat{U}(x)$, acting on some Hilbert space $\mathcal{H}$, parametrized by points x of a measure space X. $\hat{D}(x)$ and $\hat{U}(x)$ are called quantizer and dequantizer, respectively.

The bijective map $\hat{A} \leftrightarrow A(x)$ of operators onto their symbols is given by taking the trace of the product of $\hat{A}$ with the dequantizer. The reconstruction of operators from their symbols is given by integration of the product $f_A(x)\hat{D}(x)$ over the measure space X.

The kernel of the noncommutative star–product of the symbols $A(x)$ and $B(x)$ of the operators $\hat{A}$ and $\hat{B}$ is determined by the "structure constants" $K(x_1, x_2; x_3) = \mathrm{Tr}[\hat{D}(x_1)\hat{D}(x_2)\hat{U}(x_3)]$. Thus, given the quantizer $\hat{D}(x)$ and the dequantizer $\hat{U}(x)$ one obtains the star-product kernel. To the best of our knowledge the inverse problem was not considered till now. Namely, given a star-product kernel of functions on a manifold, is it possible to find a pair of quantizers-dequantizers which allows to realize the star-product kernel by the tracing formula? The aim of this paper is to obtain the explicit equation for the quantizer if the star-product kernel is given. We will show that such important example as Grönewold kernel [16] provides the equation for finding the quantizer in the scheme of Weyl–Moyal–Wigner symbols [17]. We also consider some other known and unknown examples.

The paper is organized as follows. In the next Section 2 we review the construction of the star-product scheme following [12] . In Sect. 3 the equation for the quantizer of the star-product scheme is derived. In Sect. 4 we study a known example of Moyal star-product. In Sect. 5 we apply the method to find the quantizer for a discrete spin-system. In Sect. 6 we consider the symplectic tomographic map and the equation for the tomographic product kernel. In Sect. 7 we resume our results and perspectives.

2 Quantizer–dequantizer pair and product

Symbols of operators $\hat{A}$ and $\hat{B}$, determined by the dequantizer $\hat{U}(x)$ are given by

$$\begin{aligned} f_{\hat{A}}(x) &= \mathrm{Tr}[\hat{U}(x)\hat{A}] = A(x) \\ f_{\hat{B}}(x) &= \mathrm{Tr}[\hat{U}(x)\hat{B}] = B(x) \end{aligned} \tag{1}$$

while the inverse are given by means of the quantizer $\hat{D}(x)$ as

$$\begin{aligned} \hat{A} &= \int f_{\hat{A}}(x)\hat{D}(x)dx \\ \hat{B} &= \int f_{\hat{B}}(x)\hat{D}(x)dx \end{aligned} \tag{2}$$

provided that

$$\mathrm{Tr}[\hat{U}(x)\hat{D}(x')] = \delta(x - x'). \tag{3}$$

The star-product is defined by the kernel $K(x_1, x_2, x_3)$, i.e.

$$(f_A * f_B)(x_3) = \int K(x_1, x_2; x_3) f_A(x_1) f_B(x_2) dx_1 dx_2. \tag{4}$$

The kernel itself is determined by quantizer and dequantizer as [12]

$$K(x_1, x_2; x_3) = \mathrm{Tr}\left[\hat{D}(x_1)\hat{D}(x_2)\hat{U}(x_3)\right] \tag{5}$$

on the space X. Out of this construction one obtains the bilinear, binary associative product of functions $f_A(x)$, $f_B(x)$, $f_C(x)$ i.e.

$$((f_A * f_B) * f_C)(x) = (f_A * (f_B * f_C))(x). \tag{6}$$

Property (6) means that the kernel $K(x_1, x_2; x_3)$ satisfies the nonlinear (quadratic) equation

$$\int K(x_1, x_2; t) K(t, x_3; x_4) dt = \int K(x_1, t; x_4) K(x_2, x_3; t) dt. \tag{7}$$

In [18] the study of a dual star-product scheme was considered. The pair $\hat{U}(x)$ and $\hat{D}(x)$ can be used to construct the dual symbol of an operator $\hat{A}$ by exchanging the role of the initial quantizer and dequantizer and considering the new dequantizer $\hat{U}^d(x)$ as

$$\hat{U}^d(x) = \hat{D}(x) \tag{8}$$

and the new quantizer $\hat{D}^d(x)$ as

$$\hat{D}^d(x) = \hat{U}(x). \tag{9}$$

These new operators satisfy the compatibility condition

$$\mathrm{Tr}[\hat{U}(x)\hat{D}(x')] = \mathrm{Tr}[\hat{U}^d(x)\hat{D}^d(x')] = \delta(x - x'). \tag{10}$$

In view of this, the dual symbol of an operator $\hat{A}$ reads

$$f^d_A(x) = \mathrm{Tr}[\hat{D}(x)\hat{A}] \tag{11}$$

and the reconstruction formula provides an expression for the operator $\hat{A}$ in terms of its dual symbol

$$\hat{A} = \int f^d_A(x)\hat{U}(x)dx. \tag{12}$$

The dual star-product kernel is given by the same formula (5) with the replacement $\hat{D} \leftrightarrow \hat{U}$, i.e.

$$K^d(x_1, x_2; x_3) = \mathrm{Tr}\left[\hat{U}(x_1)\hat{U}(x_2)\hat{D}(x_3)\right]. \tag{13}$$

The meaning of the dual symbols and the dual star-product is based on the possibility to express the mean value of a quantum observable $\hat{A}$ in the form analogous to the formula of standard probability theory [18, 19], i.e.

$$\langle \hat{A} \rangle = \mathrm{Tr}[\hat{\rho}\hat{A}] = \int \mathcal{W}(x)\mathcal{W}^d_A(x)dx. \tag{14}$$

If $\mathcal{W}(x)$ is the symbol of the density operator $\hat{\rho}$ and this symbol is such that it has the property of a fair probability distribution like in the tomographic picture of quantum mechanics, the dual symbol $\mathcal{W}^d_A(x)$ of an observable $\hat{A}$ plays the role of the function identified with the observable in the star-product scheme under consideration. Then the dual star-product kernel (13) provides a rule of multiplication for the observables. The Weyl–Wigner–Moyal star–product is self–dual since in this scheme $\hat{U}(x) = \lambda\hat{D}(x)$.

3 Equations for the kernel and the quantizer

We are now able to formulate the main problem of the present paper: Given the associative product with kernel $K(x_1, x_2, x_3)$, can we find the pair $\hat{U}(x)$ and $\hat{D}(x)$ which provides the kernel by means of equation (5)? We are searching for an equation for the pair $\hat{U}(x)$ and $\hat{D}(x)$. This equation can be obtained in the following way. Let us first suppose, for a given kernel, that the unknown dequantizer $\hat{D}(x)$ exists. Then let us construct the operator

$$\hat{F}(x_1, x_2) = \int K(x_1, x_2; x_3)\hat{D}(x_3)dx_3. \tag{15}$$

The kernel can be interpreted as the symbol of the operator product $\hat{D}(x_1)\hat{D}(x_2)$ if one recalls equations (1) and (5). Thus, due to the reconstruction formulae (2), one has

$$\hat{F}(x_1, x_2) = \int K(x_1, x_2; x_3)\hat{D}(x_3)dx_3 = \hat{D}(x_1)\hat{D}(x_2). \tag{16}$$

In this formula the kernel is known and the quantizer $\hat{D}(x)$ is unknown. It is just the equation which we are looking for.

Together with formula (3), equation (16) gives (in principle) the pair quantizer $\hat{D}(x)$ and dequantizer $\hat{U}(x)$.

From our analysis for a given kernel of the dual star-product $K^d(x_1, x_2, x_3)$ follows the equation for finding the operator $\hat{U}(x)$ which reads

$$\hat{U}(x_1)\hat{U}(x_2) = \int K^d(x_1, x_2; x_3)\hat{U}(x_3)dx_3. \tag{17}$$

In finite terms, we assume to have a vector space V, with a given basis $\{v_j\}$ $j = 1, \dots n$, and structure constants for an associative product $v_j \cdot v_k = \sum_l C^l_{jk} v_l$, our inverse problem amounts to find matrices D_j such that $D_j D_k = \sum_l C^l_{jk} D_l$.

In the next section we illustrate this by using the Moyal-Grönewold product and the tomographic one.

4 Solving the equation for the quantizer of known star-products

Let us check now the validity of our Eq.(16) for the Weyl product. We put $\hbar = 1$.The dequantizer for the Weyl symbol is the displaced parity operator

$$\hat{U}(q,p) \equiv \hat{U}(z) := 2\hat{\mathcal{D}}(z)\hat{\mathcal{P}}\hat{\mathcal{D}}^\dagger(z) = 2\hat{\mathcal{D}}(2z)\hat{\mathcal{P}} \ , \ z = \frac{q+ip}{\sqrt{2}}, \tag{18}$$

where $\hat{\mathcal{D}}(z) = \exp\left[z\hat{a}^\dagger - z^*\hat{a}\right]$ is the usual displacement operator and $\hat{\mathcal{P}} = \exp\left[i\pi\hat{a}^\dagger\hat{a}\right]$ is the parity operator.

The quantizer is

$$\hat{D}(q,p) := \frac{1}{2\pi}\hat{U}(q,p). \tag{19}$$

Using $z_k = (q_k + ip_k)/\sqrt{2}, k = 1,2,3$, the equation for the quantizer determined by the Grönewold kernel may be put in the form

$$\begin{aligned} & \exp\left[2(z_1^* z_2 - z_1 z_2^*)\right] \int \exp\left[2z_3^*\left(z_1 - z_2\right) - 2z_3\left(z_1 - z_2\right)^*\right] \hat{\mathcal{D}}(2z_3)\frac{dq_3 dp_3}{\pi}\hat{\mathcal{P}} \\ = \ & \hat{\mathcal{D}}(2z_1)\hat{\mathcal{P}}\hat{\mathcal{D}}(2z_2)\hat{\mathcal{P}}. \end{aligned} \tag{20}$$

The integral above is the complex Fourier transform of the displacement operator, which is known to be the displaced parity operator (see, e.g., eq.s (2.14) and (4.11) of [20]). So, the l.h.s. of the above equation becomes

$$\exp\left[2(z_1^* z_2 - z_1 z_2^*)\right] \hat{\mathcal{D}}(2\left[z_1 - z_2\right])\hat{\mathcal{P}}^2 = \hat{\mathcal{D}}(2z_1)\hat{\mathcal{D}}(-2z_2)\hat{\mathcal{P}}^2 = \hat{\mathcal{D}}(2z_1)\hat{\mathcal{P}}\hat{\mathcal{D}}(2z_2)\hat{\mathcal{P}}. \tag{21}$$

This completes the check that eq.(16) for the Moyal product kernel provides the quantizer $\hat{D}(q,p)$ as solution.

5 The case of discrete systems

Now we check the validity of Eq. (16) for one of the star-product schemes with spin variables [18]. Let us considere four Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{22}$$

We recall their commutation relations

$$[\sigma_0, \sigma_j] = 0, \qquad [\sigma_j, \sigma_k] = 2\sum_{m=1}^{3} i\epsilon_{jkm}\sigma_m. \tag{23}$$

The associative product reads

$$\sigma_j\sigma_0 = \sigma_0\sigma_j = \sigma_j, \tag{24}$$

$$\sigma_j\sigma_k = \delta_{jk}\sigma_0 + \sum_{m=1}^{3} i\epsilon_{jkm}\sigma_m. \tag{25}$$

Following [18] in this section we use the pairing by the rule $\langle\cdot,\cdot\rangle = 2\mathrm{Tr}\,[\cdot\cdot]$, thus define the dequantizer $\hat{U}(x)$ for discrete label $x = \{0,1,2,3\}$ as the set of four matices $\hat{U}_x$

$$\left\{\hat{U}_0 = \frac{1}{2}\sigma_0, \hat{U}_1 = \frac{1}{2}\sigma_1, \hat{U}_2 = \frac{1}{2}\sigma_2, \hat{U}_3 = \frac{1}{2}\sigma_3\right\}. \tag{26}$$

The quantizer we define as $\hat{D}_x = \hat{U}_x$. One has

$$\left\langle \hat{U}_j | \hat{D}_k \right\rangle = 2\mathrm{Tr}[\hat{U}_j\hat{D}_k] = \delta_{jk}. \tag{27}$$

The kernel of the star-product reads

$$K(j,k,m) = \frac{1}{4}\mathrm{Tr}[\sigma_j\sigma_k\sigma_m]. \tag{28}$$

This kernel can be represented in the form of four matrices, namely

$$\begin{aligned}(K_0)_{jk} &= K(j,k,0), \quad (K_1)_{jk} = K(j,k,1),\\ (K_2)_{jk} &= K(j,k,2), \quad (K_3)_{jk} = K(j,k,3).\end{aligned} \tag{29}$$

One can easily get these matrices in explicit form

$$K_0 = \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \qquad K_1 = \frac{1}{2}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&i\\0&0&-i&0\end{pmatrix}, \tag{30}$$

$$K_2 = \frac{1}{2}\begin{pmatrix}0&0&1&0\\0&0&0&-i\\1&0&0&0\\0&i&0&0\end{pmatrix}, \qquad K_3 = \frac{1}{2}\begin{pmatrix}0&0&0&1\\0&0&i&0\\0&-i&0&0\\1&0&0&0\end{pmatrix},$$

Of course, a solution for eq. (16) is provided by the Pauli matrices, indeed they satisfy the condition

$$\left(\frac{1}{2}\sigma_j\right)\left(\frac{1}{2}\sigma_k\right) = \sum_{s=0}^{3}(K_s)_{jk}\frac{\sigma_s}{2}. \tag{31}$$

For example for $j = 1,\ k = 2$ from eq. (25) we get

$$\frac{1}{2}\sigma_1 \cdot \frac{1}{2}\sigma_2 = \frac{i\sigma_3}{4}. \tag{32}$$

Another solution is provided by matrices K_0, K_1, K_2, K_3 defined in eq.(30). Thus we have provided two solutions for the equation (16) for the quantizer matrices $\{D_j\}$. So, in conclusion, given the structure constants, we search for the quantizer matrices D_1, D_2, D_3, D_4 which would satisfy

$$D_j D_k = \sum_l C^l_{jk} D_l. \tag{33}$$

To give an example where the structure constants are not primarily given by a standard row–by–column product of matrices, we consider the following product on $2 \times 2-$matrices [21]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bd' \\ ca' + dc' & dd' \end{pmatrix}, \tag{34}$$

one can check that this product is associative. Then, introducing the Weyl basis of matrices (instead of Pauli matrices)

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{35}$$

we obtain the rule of multiplication

$$e_j \cdot e_k = \sum_{l=1}^{4} C^l_{jk} e_l \tag{36}$$

where, as one can see, only six components of the structure constants are non zero, i.e.

$$C^1_{11} = C^2_{12} = C^2_{24} = C^3_{31} = C^3_{43} = C^4_{44} = 1. \tag{37}$$

Introducing functions on the four dimensional linear space in the form of a $4-$vectors

$$\vec{f} = (f^1, f^2, f^3, f^4)$$

where for any abstract vector $v = \sum_{j=1}^4 v^j e_j$ the function $\vec{f}(v) = \sum_{j=1}^4 v^j \vec{f}(e_j)$ and $(\vec{f}(e_j))^k = \delta^k_j$ one has the star–product multiplication rule for the functions $\vec{f}_1$ and $\vec{f}_2$

$$(\vec{f}_1 * \vec{f}_2)^l = \sum_{j,k=1}^{4} f^j_1 C^l_{jk} f^k_2. \tag{38}$$

The quantizer $4\times 4-$matrices are defined as $(D_\gamma)^\alpha_\beta = C^\alpha_{\gamma\beta}, (\alpha,\beta,\gamma=1,2,3,4)$, and read

$$D_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{39}$$

$$D_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad D_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The corresponding dequantizer matrices can be chosen solving the duality condition

$$\mathrm{Tr}[D_j U_k] = \delta_{jk}. \tag{40}$$

The quantizers and dequantizers must close on subalgebras of the general linear group. Now one can see that by choosing

$$U_{1,4} = \frac{1}{2} D^T_{1,4} \quad , \quad U_{2,3} = D^T_{2,3} \tag{41}$$

one can check that while the D's close on an algebra with structure constants C^l_{jk}, the U's close on an algebra with structure constants $d^l_{jk} = \frac{1}{2} C^l_{kj}$. Then we found for the considered exotic rule of multiplication of matrices the corresponding star–product scheme with quantizers and dequantizers.

The considered example provides the star–product scheme with the kernel C^l_{jk} given by the standard formula

$$C^l_{jk} = \mathrm{Tr}[D_j D_k U_l] \tag{42}$$

with quantizer (39,40) and dequantizer given by (41).

A last example is given by considering the associative so–called $\kappa-$star–product [12, 22], with the matrix multiplication rule $a \circ b = a\kappa b$. In case of $2\times 2-$matrices, by choosing a Hermitian matrix κ, we may write

$$\kappa = \sum_{\alpha=0}^{3} s^\alpha \sigma_\alpha \tag{43}$$

where the components $s^\alpha, \alpha = 0,1,2,3$, are real, and the σ_α are the previous Pauli matrices with the identity σ_0.

The structure constants , with $\alpha = 0,1,2,3$, and $j,m,n = 1,2,3$, are:

$$
\begin{aligned}
C^{\alpha}_{00} &= s^{\alpha}, C^{\alpha}_{0j} = \left(C^{\alpha}_{j0}\right)^* = \delta^{\alpha}_{0} s^j + \delta^{\alpha}_{j} s^0 + \delta^{\alpha}_{m} \sum_{n=1}^{3} i s^n \epsilon_{njm}, \qquad (44)\\
C^{\alpha}_{jm} &= \delta^{\alpha}_{0}\left(s^0 \delta_{jm} + \sum_{n=1}^{3} i s^n \epsilon_{nmj}\right) + \delta^{\alpha}_{j} s^m + \delta^{\alpha}_{m} s^j + \delta^{\alpha}_{n}\left(i\ s^0 \epsilon_{jmn} - \delta_{jm} s^n\right).
\end{aligned}
$$

They give rise to the quantizer family:

$$
D_0 = \begin{pmatrix} s^0 & s^1 & s^2 & s^3 \\ s^1 & s^0 & is^3 & -is^2 \\ s^2 & -is^3 & s^0 & is^1 \\ s^3 & is^2 & -is^1 & s^0 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} s^1 & s^0 & -is^3 & is^2 \\ s^0 & s^1 & -s^2 & -s^3 \\ -is^3 & s^2 & s^1 & is^0 \\ is^2 & s^3 & -is^0 & s^1 \end{pmatrix}, \qquad (45)
$$

$$
D_2 = \begin{pmatrix} s^2 & is^3 & s^0 & -is^1 \\ is^3 & s^2 & s^1 & -is^0 \\ s^0 & -s^1 & s^2 & -s^3 \\ -is^1 & is^0 & s^3 & s^2 \end{pmatrix}, \qquad D_3 = \begin{pmatrix} s^3 & -is^2 & is^1 & s^0 \\ -is^2 & s^3 & is^0 & s^1 \\ is^1 & -is^0 & s^3 & s^2 \\ s^0 & -s^1 & -s^2 & s^3 \end{pmatrix}.
$$

The dequantizers may be found by solving an equation like (40). We conclude by observing that in the limit $\kappa \to \sigma_0$, i. e. $s^0 \to 1, s^1, s^2, s^3 \to 0$, the above matrices yield just the matrices K's of eq. (30)

6 Symplectic tomography

Now we prove that for the symplectic tomographic star-product the quantizer and dequantizer satisfy the condition of compatibility for homogeneous functions $f(X,\mu,\nu)$. In fact the quantizer $\hat{D}(X,\mu,\nu)$ and dequantizer $\hat{U}(X,\mu,\nu)$, say

$$
\hat{D}(X,\mu,\nu) = \frac{1}{2\pi} \exp i(X - \mu\hat{q} - \nu\hat{p}), \hat{U}(X,\mu,\nu) = \delta(X - \mu\hat{q} - \nu\hat{p}), \qquad (46)
$$

give

$$
\mathrm{Tr}\left[\hat{U}(1)\hat{D}(2)\right] = \frac{1}{2\pi} \mathrm{Tr}\left[\delta(X_1 - \mu_1\hat{q} - \nu_1\hat{p}) \exp i(X_2 - \mu_2\hat{q} - \nu_2\hat{p})\right], \qquad (47)
$$

which can be expressed by its action on homogeneous functions as

$$
\frac{1}{(2\pi)^2} \int \mathrm{e}^{i(X' - kX)} \delta(\mu' - k\mu) \delta(\nu' - k\nu) f(X',\mu',\nu') dX' d\mu' d\nu'. \qquad (48)
$$

Taking the Fourier transform with respect to the variable X' one has the expression

$$\frac{1}{2\pi}\int \tilde{f}(-1,-k\mu,-k\nu)e^{ikX}dk = f(X,\mu,\nu). \tag{49}$$

We used the property of the Fourier transform of the homogeneous tomographic symbol

$$\tilde{f}(k,\lambda\mu,\lambda\nu) = \tilde{f}(k\lambda,\mu,\nu) \tag{50}$$

Thus we proved that the action of $\mathrm{Tr}[\hat{U}(x)\hat{D}(x')]$ onto the function $f(x), x = (X,\mu,\nu)$, which is homogeneous of degree -1, is equivalent to the integration of this function with the Dirac delta-function $\delta(x-x')$. However, the integration with this function of non homogenous functions $F(X',\mu',\nu')$ does not provide the same function $F(X,\mu,\nu)$. This property takes place also if we consider the solution of the equation for the finding the quantizer of the tomographic star–product scheme. In fact the relation

$$\int \delta(\nu_3(\mu_1+\mu_2)-\mu_3(\nu_1+\nu_2))\exp\left[-i\frac{\nu_1+\nu_2}{\nu_3}X_3\right]\hat{D}(X_3,\mu_3,\nu_3)dX_3d\mu_3d\nu_3$$
$$\times\frac{1}{4\pi^2}\exp\left[iX_1+iX_2+\frac{i}{2}(\nu_1\mu_2-\nu_2\mu_1)\right] = \hat{D}(X_1,\mu_1,\nu_1)\hat{D}(X_2,\mu_2,\nu_2) \tag{51}$$

holds true if one applies these operators to homogeneous functions.

7 Conclusions

We summarize the main results of our paper. Given the kernel of a star-product which provides an associative product of functions on some measure space X, is it possible to find a Hilbert space and a pair of operator families, labeled by points of the space and called quantizer and dequantizer, such that the kernel of the star–product is obtained by tracing the product of two quantizers and one dequantizer? The answer which we obtained is affirmative. The solution is provided by a nonlinear equation, eq. (16), for the quantizer operators for any given product kernel.

We checked on the examples of the Moyal product, the tomographic product, and the products defined on functions depending on discrete spin-variables that there always exist solutions for the obtained equation for quantizers. We conjecture that this situation takes place for arbitrary star-products of functions both in finite and infinite spaces. In some sense we would generalize the known result of Gelfand–Naimark–Segal which asserts that one can always construct (GNS construction) a Hilbert space and the operators which give a representation of a given $C^{\star}-$algebra. We conjecture that all star–products

on a measure space can be realized by means of quantizers and dequantizers by the construction we have considered. Our result may be considered also as an extension to associative algebras of Ado's theorem available in the setting of Lie algebras, assuring that any Lie product for abstract finite dimensional Lie algebra may be realized as the commutator of matrices.

In a future paper we shall consider the method of contraction of associative algebras by using a contraction procedure on quantizers and dequantizers.

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